# 11-2

# Arithmetic Series

## Main Ideas

- Find sums of arithmetic series.
- Use sigma notation.

## New Vocabulary

series
arithmetic series
sigma notation
index of summation

## GET READY for the Lesson

Austin, Texas has a strong musical tradition. It is home to many indoor and outdoor music venues where new and established musicians perform regularly. Some of these venues are amphitheaters that generally get wider as the distance from the stage increases.

Suppose a section of an amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.

**Arithmetic Series** The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is 18 + 22 + 26 + 30 or 96. A **series** is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, 18 + 22 + 26 + 30 is an **arithmetic series**.

### Study Tip

**Indicated Sum**

The sum of a series is the result when the terms of the series are added. An *indicated sum* is the expression that illustrates the series, which includes the terms + or −.

$S_n$ represents the sum of the first $n$ terms of a series. For example, $S_4$ is the sum of the first four terms.

To develop a formula for the sum of any arithmetic series, consider the series below.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

Write $S_9$ in two different orders and add the two equations.

$$\begin{array}{rl} S_9 = & 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60 \\ (+)\ S_9 = & 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4 \\ \hline 2S_9 = & 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 \\ 2S_9 = & 9(64) \\ S_9 = & \frac{9}{2}(64) \end{array}$$

Note that the sum had 9 terms.

The sum of the first and last terms of the series is 64.

An arithmetic series $S_n$ has $n$ terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.

Allen Matheson/photohome.com

### KEY CONCEPT — Sum of an Arithmetic Series

The sum $S_n$ of the first $n$ terms of an arithmetic series is given by

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \text{ or } S_n = \frac{n}{2}(a_1 + a_n).$$

## EXAMPLE Find the Sum of an Arithmetic Series

**1** **Find the sum of the first 100 positive integers.**

The series is $1 + 2 + 3 + \dots + 100$. Since you can see that $a_1 = 1$, $a_{100} = 100$, and $d = 1$, you can use either sum formula for this series.

| **Method 1** | | **Method 2** |
|---|---|---|
| $S_n = \frac{n}{2}(a_1 + a_n)$ | Sum formula | $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ |
| $S_{100} = \frac{100}{2}(1 + 100)$ | $n = 100, a_1 = 1,$<br>$a_{100} = 100, d = 1$ | $S_{100} = \frac{100}{2}[2(1) + (100 - 1)1]$ |
| $S_{100} = 50(101)$ | Simplify. | $S_{100} = 50(101)$ |
| $S_{100} = 5050$ | Multiply. | $S_{100} = 5050$ |

### CHECK Your Progress

**1.** Find the sum of the first 50 positive even integers.

## Real-World EXAMPLE Find the First Term

**Real-World Link**

99.0% of teens ages 12–17 listen to the radio at least once a week. 79.1% listen at least once a day.

**Source:** Radio Advertising Bureau

**2** **RADIO** **A radio station is giving away a total of \$124,000 in August. If they increase the amount given away each day by \$100, how much should they give away the first day?**

You know the values of $n$, $S_n$, and $d$. Use the sum formula that contains $d$.

| | |
|---|---|
| $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ | Sum formula |
| $S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100]$ | $n = 31, d = 100$ |
| $124{,}000 = \frac{31}{2}(2a_1 + 3000)$ | $S_{31} = 124{,}000$ |
| $8000 = 2a_1 + 3000$ | Multiply each side by $\frac{2}{31}$. |
| $5000 = 2a_1$ | Subtract 3000 from each side. |
| $2500 = a_1$ | Divide each side by 2. |

The radio station should give away \$2500 the first day.

### CHECK Your Progress

**2.** **EXERCISE** Aiden did pushups every day in March. He started on March 1st and increased the number of pushups done each day by one. He did a total of 1085 pushups for the month. How many pushups did Aiden do on March 1st?

**Online** Personal Tutor at algebra2.com

Michelle Bridwell/PhotoEdit

Sometimes it is necessary to use both a sum formula and the formula for the $n$th term to solve a problem.

### EXAMPLE Find the First Three Terms

**3** **Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.**

**Step 1** Since you know $a_1$, $a_n$, and $S_n$, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find $n$.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$741 = \frac{n}{2}(9 + 105)$$

$$741 = 57n$$

$$13 = n$$

**Step 2** Find $d$.

$$a_n = a_1 + (n - 1)d$$

$$105 = 9 + (13 - 1)d$$

$$96 = 12d$$

$$8 = d$$

**Step 3** Use $d$ to determine $a_2$ and $a_3$.

$a_2 = 9 + 8$ or $17$ $\quad$ $a_3 = 17 + 8$ or $25$

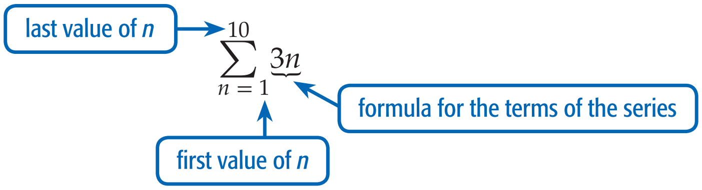
The first three terms are 9, 17, and 25.

**CHECK Your Progress**

**3.** Find the first three terms of an arithmetic series in which $a_1 = -16$, $a_n = 33$, and $S_n = 68$.

**Sigma Notation** Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series $3 + 6 + 9 + 12 + \cdots + 30$ can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read *the sum of* 3*n as n goes from 1 to 10.*

**Concepts in Motion**
**Animation**
algebra2.com



The variable, in this case $n$, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of $n$ are 1, 2, 3, and so on, through 10.

There are many ways to represent a given series. If changes are made to the first and last values of the variable and to the formula for the terms of the series, the same terms can be produced. For example, the following expressions produce the same terms.

$$\sum_{r=4}^{9} (r - 3) \qquad \sum_{s=2}^{7} (s - 1) \qquad \sum_{j=0}^{5} (j + 1)$$

**Math Online** **Extra Examples at** algebra2.com

## EXAMPLE Evaluate a Sum in Sigma Notation

**4** Evaluate $\sum_{j=5}^{8}(3j - 4)$.

### Method 1

Find the terms by replacing $j$ with 5, 6, 7, and 8. Then add.

$$\begin{aligned}\sum_{j=5}^{8}(3j-4) &= [3(5)-4]+[3(6)-4]+[3(7)-4]+[3(8)-4]\\ &= 11+14+17+20\\ &= 62\end{aligned}$$

### Method 2

Since the sum is an arithmetic series, use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

There are 4 terms, $a_1 = 3(5) - 4$ *or* 11, and $a_4 = 3(8) - 4$ or 20.

$$\begin{aligned}S_4 &= \frac{4}{2}(11+20)\\ &= 62\end{aligned}$$

### CHECK Your Progress

**4.** Evaluate $\sum_{k=2}^{6}(2k + 1)$.

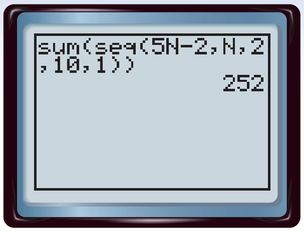
You can use the sum and sequence features on a graphing calculator to find the sum of a series.

## GRAPHING CALCULATOR LAB

### Sums of Series

The calculator screen shows the evaluation of $\sum_{N=2}^{10}(5N - 2)$. The first four entries for seq( are

- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.



```
sum(seq(5N-2,N,2
,10,1))
             252
```

The last entry is always 1 for the types of series that we are considering.

**THINK AND DISCUSS**

**1.** Explain why you can use any letter for the index of summation.

**2.** Evaluate $\sum_{n=1}^{8}(2n - 1)$ and $\sum_{j=5}^{12}(2j - 9)$. Make a conjecture as to their relationship and explain why you think it is true.

**Study Tip**

**Graphing Calculators**

On the TI-83/84 Plus, sum( is located on the LIST MATH menu. The function seq( is located on the LIST OPS menu.

## CHECK Your Understanding

**Example 1**
**(p. 630)**

**Find the sum of each arithmetic series.**

**1.** $5 + 11 + 17 + \cdots + 95$

**2.** $12 + 17 + 22 + \cdots + 102$

**3.** $38 + 35 + 32 + \cdots + 2$

**4.** $101 + 90 + 79 + \cdots + 2$

**5.** **TRAINING** To train for a race, Rosmaria runs 1.5 hours longer each week than she did the previous week. In the first week, Rosmaria ran 3 hours. How much time will Rosmaria spend running if she trains for 12 weeks?

**Examples 1, 2**
**(p. 630)**

**Find $S_n$ for each arithmetic series described.**

**6.** $a_1 = 4, a_n = 100, n = 25$

**7.** $a_1 = 40, n = 20, d = -3$

**8.** $d = -4, n = 21, a_n = 52$

**9.** $d = 5, n = 16, a_n = 72$

**Example 2**
**(p. 630)**

**Find $a_1$ for each arithmetic series described.**

**10.** $d = 8, n = 19, S_{19} = 1786$

**11.** $d = -2, n = 12, S_{12} = 96$

**Example 3**
**(p. 631)**

**Find the first three terms of each arithmetic series described.**

**12.** $a_1 = 11, a_n = 110, S_n = 726$

**13.** $n = 8, a_n = 36, S_n = 120$

**Example 4**
**(p. 632)**

**Find the sum of each arithmetic series.**

**14.** $\sum_{n=1}^{7} (2n + 1)$

**15.** $\sum_{k=3}^{7} (3k + 4)$

## Exercises

**HOMEWORK HELP**

| For Exercises | See Examples |
|---|---|
| 16–21, 34–37 | 1 |
| 22–25 | 1, 2 |
| 26–29 | 2 |
| 30–33 | 3 |
| 38–43 | 4 |

**Find $S_n$ for each arithmetic series described.**

**16.** $a_1 = 7, a_n = 79, n = 8$

**17.** $a_1 = 58, a_n = -7, n = 26$

**18.** $a_1 = 7, d = -2, n = 9$

**19.** $a_1 = 3, d = -4, n = 8$

**20.** $a_1 = 5, d = \frac{1}{2}, n = 13$

**21.** $a_1 = 12, d = \frac{1}{3}, n = 13$

**22.** $d = -3, n = 21, a_n = -64$

**23.** $d = 7, n = 18, a_n = 72$

**24.** **TOYS** Jamila is making a wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?

**25.** **CONSTRUCTION** A construction company will be fined for each day it is late completing a bridge. The daily fine will be \$4000 for the first day and will increase by \$1000 each day. Based on its budget, the company can only afford \$60,000 in total fines. What is the maximum number of days it can be late?

**Find $a_1$ for each arithmetic series described.**

**26.** $d = 3.5, n = 20, S_{20} = 1005$

**27.** $d = -4, n = 42, S_{42} = -3360$

**28.** $d = 0.5, n = 31, S_{31} = 573.5$

**29.** $d = -2, n = 18, S_{18} = 18$

**EXTRA PRACTICE**
See pages 914, 936.

**Math Online**
Self-Check Quiz at
algebra2.com

**Find the first three terms of each arithmetic series described.**

**30.** $a_1 = 17, a_n = 197, S_n = 2247$

**31.** $a_1 = -13, a_n = 427, S_n = 18{,}423$

**32.** $n = 31, a_n = 78, S_n = 1023$

**33.** $n = 19, a_n = 103, S_n = 1102$

**Find the sum of each arithmetic series.**

**34.** $6 + 13 + 20 + 27 + \cdots + 97$

**35.** $7 + 14 + 21 + 28 + \cdots + 98$

**36.** $34 + 30 + 26 + \cdots + 2$

**37.** $16 + 10 + 4 + \cdots + (-50)$

**38.** $\sum_{n=1}^{6} (2n + 11)$

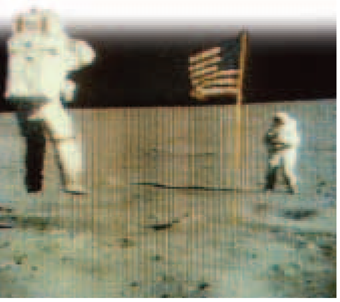
**39.** $\sum_{n=1}^{5} (2 - 3n)$

**40.** $\sum_{k=7}^{11} (42 - 9k)$

**41.** $\sum_{t=19}^{23} (5t - 3)$

**42.** $\sum_{n=1}^{300} (7n - 3)$

**43.** $\sum_{k=1}^{150} (11 + 2k)$



**Real-World Link**

Six missions of the Apollo Program landed humans on the Moon. Apollo 11 was the first mission to do so.

**Source:** nssdc.gsfc.nasa.gov

**Find $S_n$ for each arithmetic series described.**

**44.** $a_1 = 43, n = 19, a_n = 115$

**45.** $a_1 = 76, n = 21, a_n = 176$

**46.** $a_1 = 91, d = -4, a_n = 15$

**47.** $a_1 = -2, d = \frac{1}{3}, a_n = 9$

**48.** $d = \frac{1}{5}, n = 10, a_n = \frac{23}{10}$

**49.** $d = -\frac{1}{4}, n = 20, a_n = -\frac{53}{12}$

**50.** Find the sum of the first 1000 positive even integers.

**51.** What is the sum of the multiples of 3 between 3 and 999, inclusive?

**52.** **AEROSPACE** On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than it did the previous second. How far would an object fall in the first ten seconds after being dropped?

**53.** **SALARY** Mr. Vacarro's salary this year is $41,000. If he gets a raise of $2500 each year, how much will Mr. Vacarro earn in ten years?

**Graphing Calculator**

**Use a graphing calculator to find the sum of each arithmetic series.**

**54.** $\sum_{n=21}^{75} (2n + 5)$

**55.** $\sum_{n=10}^{50} (3n - 1)$

**56.** $\sum_{n=20}^{60} (4n + 3)$

**57.** $\sum_{n=17}^{90} (1.5n + 13)$

**58.** $\sum_{n=22}^{64} (-n + 70)$

**59.** $\sum_{n=26}^{50} (-2n + 100)$

**H.O.T. Problems**

**60.** **OPEN ENDED** Write an arithmetic series for which $S_5 = 10$.

**CHALLENGE** **State whether each statement is *true* or *false*. Explain your reasoning.**

**61.** Doubling each term in an arithmetic series will double the sum.

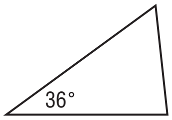
**62.** Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.

**63.** ***Writing in Math*** Use the information on page 629 to explain how arithmetic series apply to amphitheaters. Explain what the sequence and the series that can be formed from the given numbers represent, and show two ways to find the seating capacity of the amphitheater if it has ten rows of seats.

NASA

## STANDARDIZED TEST PRACTICE

**64. ACT/SAT** The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is 36°, what is the measure of the largest angle?



**A** 75° **B** 84° **C** 90° **D** 97°

**65. REVIEW** How many 5-inch cubes can be placed completely inside a box that is 10 inches long, 15 inches wide, and 5 inches tall?

**F** 5 **H** 20

**G** 6 **J** 15

## Spiral Review

**Find the indicated term of each arithmetic sequence.** (Lesson 11-1)

**66.** $a_1 = 46, d = 5, n = 14$

**67.** $a_1 = 12, d = -7, n = 22$

**Solve each system of inequalities by graphing.** (Lesson 10-7)

**68.** $9x^2 + y^2 < 81$
$x^2 + y^2 \geq 16$

**69.** $(y - 3)^2 \geq x + 2$
$x^2 \leq y + 4$

**Write an equivalent logarithmic equation.** (Lesson 9-2)

**70.** $5^x = 45$

**71.** $7^3 = x$

**72.** $b^y = x$

**73. PAINTING** Two employees of a painting company paint houses together. One painter can paint a house alone in 3 days, and the other painter can paint the same size house alone in 4 days. How long will it take them to paint one house if they work together? (Lesson 8-6)

**Simplify.** (Lesson 7-5)

**74.** $5\sqrt{3} - 4\sqrt{3}$

**75.** $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$

**76.** $\left(\sqrt{10} - \sqrt{6}\right)\left(\sqrt{5} + \sqrt{3}\right)$

**Solve each equation by completing the square.** (Lesson 5-5)

**77.** $x^2 + 9x + 20.25 = 0$

**78.** $9x^2 + 96x + 256 = 0$

**79.** $x^2 - 3x - 20 = 0$

**Use a graphing calculator to find the value of each determinant.** (Lesson 4-5)

**80.** $\begin{vmatrix} 1.3 & 7.2 \\ 6.1 & 5.4 \end{vmatrix}$

**81.** $\begin{vmatrix} 6.1 & 4.8 \\ 9.7 & 3.5 \end{vmatrix}$

**82.** $\begin{vmatrix} 8 & 6 & -5 \\ 10 & -7 & 3 \\ 9 & 14 & -6 \end{vmatrix}$

**Solve each system of equations by using either substitution or elimination.** (Lesson 3-2)

**83.** $a + 4b = 6$
$3a + 2b = -2$

**84.** $10x - y = 13$
$3x - 4y = 15$

**85.** $3c - 7d = -1$
$2c - 6d = -6$

## GET READY for the Next Lesson

**PREREQUISITE SKILL** **Evaluate the expression $a \cdot b^{n-1}$ for the given values of $a$, $b$, and $n$.** (Lesson 1-1)

**86.** $a = 1, b = 2, n = 5$

**87.** $a = 2, b = -3, n = 4$

**88.** $a = 18, b = \frac{1}{3}, n = 6$